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MOTION OF A FLEXIBLE FINITE-LENGTH FILAMENT IN A FLOW OF A VISCOUS FLUID

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The two-dimensional problem of the configuration of a flexible filament of finite length in a deformable viscous fluid is solved. The flexural stresses in the filament and the inertial and gravitational forces are not taken into account. The equilibrium equations are obtained. The friction force that acts on the filament surface from the side of the viscous fluid is proportional to the flow rate. The specific features of the evolution of a bent filament under the conditions of pure and simple shear of a fluid are studied numerically. Analytical solutions are obtained for the evolution of a rectilinear filament; in particular, the stretching force in the filament is found. For the indicated types of flow, the stability of a rectilinear filament against small perturbations is investigated.

One of the problems of producing reinforced polymer compositions is mixing of filaments with a polymer matrix. During mixing the forces that appear in the filaments are such that a filler from steel wire is torn, and the glass filaments become a dust [1–3]. This reduces the strength of articles. In addition, there is no theoretical explanation of the so-called *calender effect*, which is manifested in the anisotropy of the strength properties of chaotically reinforced polymer compositions. The effect is due to the orientation of a filler of the filaments along the direction of calendering. An experimental study of the dynamics of a separate filament and measurements of the stretching forces involve significant technical difficulties. The author failed to find a theoretical treatment of this process in the literature.

The theoretical studies of the dynamic effect of a fluid (gas) flow on a flexible filament performed by Kochin [4] and Krylov [5] are noteworthy. Kochin [4] solved the problem of the change in the shape of the kite balloon's hawser under the action of a wind. Krylov's problem of the equilibrium conditions of a spherical mine in flow is similar [5]. The dynamic effect of the flow of a viscous fluid on the filler's fiber protruded like a cantilever is analyzed by Kim and Skachkov [6]. To determine the elastic line of a fiber, it was proposed to partition the length of the fiber into a finite number of segments, within the limits of which the fiber rigidity is constant, and the external loads (forces and moments) act only at the ends of the segments.

The aim of the present study is to consider theoretically the evolution of the shape and tension of an inextensible flexible filament of finite length under the conditions of pure shear and of a Couette plane flow of a viscous fluid, analyze the mechanism of the "orienting" effect of a fluid on a rectilinear filament, and study the stability of a rectilinear filament against small perturbations.

1. Formulation of the Problem. The velocity field in mixers is generally three-dimensional, but we confine ourselves to the consideration of a two-dimensional flow to clarify the main specific features (evolution of the shape and tension). For example, the velocity field in mixing rollers is two-dimensional. In this case,

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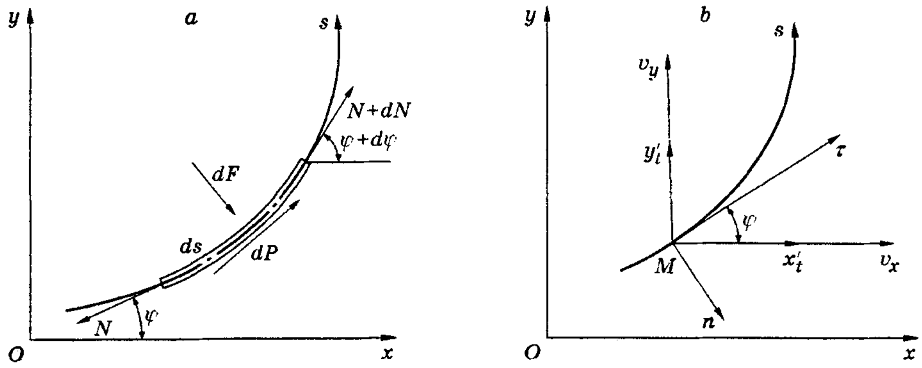

Fig. 1

the simple shear dominates near the roller surface, whereas the material deformation is close to pure shear in the middle zone of the flow [7].

An ideal flexible filament in a flow of a viscous fluid is considered (Fig. 1a). Since this filament does not offer resistance to bending, the internal force is only the tension N which acts along the tangent to the filament axis. The inertial and gravitational forces are negligible compared to the axial tension. The filament does not contact with other filaments. It is affected by the friction force from the side of the deformable viscous fluid, but the velocity field in the fluid remains unchanged. The elastic strains connected with expansion or compression of the filament are not taken into account. There are no segments of large (or infinite) curvature on the filament. The flow is laminar and isothermal. The filament axis remains a flat curve.

Let us construct the equilibrium equations. In the Cartesian coordinates xOy, the configuration of the filament axis in parametric form is described by the functions $x(s)$ and $y(s)$, where s is the coordinate reckoned along the filament axis. This axis lies in the xOy plane. The filament's element of length ds is subjected to the friction force whose projection is dF on the normal and dP on the tangent (Fig. 1a). The angle between the horizontal direction, which we assume to be the direction of the x axis and the tangent to the filament axis, is denoted by φ. Then, the equilibrium equations (the Kirchhoff equations) have the form

$$\Sigma X = 0\colon\quad (N + dN)\cos(\varphi + d\varphi) - N\cos\varphi + dF\sin(\varphi + d\varphi/2) + dP\cos(\varphi + d\varphi/2) = 0,$$

$$\Sigma Y = 0\colon\quad (N + dN)\sin(\varphi + d\varphi) - N\sin\varphi + dP\sin(\varphi + d\varphi/2) - dF\cos(\varphi + d\varphi/2) = 0.$$

With allowance for the relations

$$\cos(\varphi + d\varphi) = \cos\varphi - d\varphi\,\sin\varphi + \ldots, \qquad \sin(\varphi + d\varphi) = \sin\varphi + d\varphi\cos\varphi + \ldots,$$

ignoring infinitesimals that have an order of magnitude higher than the first one, we obtain the following equations:

$$d(Nx') + y'\,dF + x'\,dP = 0; \tag{1.1}$$

$$d(Ny') + y'\,dP - x'\,dF = 0. \tag{1.2}$$

The relations $\sin\varphi = dy/ds = y'$ and $\cos\varphi = dx/ds = x'$ were taken into account. Hereafter, the derivative with respect to s is primed.

We differentiate the coupling equation (inelasticity condition for the filament axis)

$$(x')^2 + (y')^2 = 1 \tag{1.3}$$

with respect to s:

$$x''x' + y''y' = 0. \tag{1.4}$$

We multiply Eq. (1.1) by x' and Eq. (1.2) by y'; adding them, with allowance for (1.3) and (1.4) we have

$$dN + dP = 0. \tag{1.5}$$

Similarly, multiplying Eq. (1.1) by y' and (1.2) by x' and subtracting the resulting equalities, with allowance for relations (1.3) and (1.4) we obtain the second equilibrium equation

$$F' - Ny''/x' = 0. \tag{1.6}$$

The equations of evolution of the shape of a flat filament are derived with allowance for the dependence of the friction force on the fluid and filament velocities in (1.5) and (1.6). In the inertialess approximation, a change in the friction force caused by fluid deformation causes a simultaneous change in the filament configuration and tension.

Let us consider the components of the friction force. We assume that the filament is moistened with a fluid and that the adhesion condition is satisfied. The minimum radius of curvature is much greater than the filament diameter. There is a three-dimensional boundary layer near the surface of the moving filament. By virtue of the linearity of the Stokes equations (the inertialess approximation for small Reynolds numbers), the motion in the boundary layer can be regarded as a superposition of two motions: transverse and longitudinal flows about the filament. It is noteworthy that, for non-Newtonian fluids, the principle of superposition of the flows is not satisfied.

For a transverse flow about an infinite cylinder, the friction force is determined by the Lamb formula [8]

$$dF = B_n \Delta v_n \, ds, \tag{1.7}$$

where $B_n = 4\pi\mu/\ln(7.4/\mathrm{Re})$, $\mathrm{Re} = \langle v\rangle d\rho/\mu$ is the Reynolds number, μ is the viscosity of the fluid; d is the filament diameter, ρ is the density of the fluid, Δv_n is the relative velocity of the transverse flow, and $\langle v\rangle$ is the characteristic velocity.

In formula (1.7), the coefficient B_n takes into account the effect of the filament diameter. This coefficient changes slightly over a broad interval of Reynolds numbers (as Re increases from 10^{-8} to 10^{-3}, the coefficient B_n increases by a factor of 2.3). Therefore, the value of B_n is assumed to be constant and to correspond to the characteristic velocity of the fluid flow about the filament.

When the systems filled by filaments are mixed, the fluid flow about a separate filament is topologically limited to the hydrodynamic influence of the adjacent filaments, which form a tube of the kind around it. Therefore, in the first approximation, we consider a filament that moves axially in a cylindrical tube filled with a viscous fluid (axisymmetric Couette flow). The radius of a conventional tube $\langle r\rangle$ is determined by the average distance between the filaments and is related to their volumetric concentration $\langle c\rangle$ by the relation $\langle r\rangle = d/(2.1\sqrt{\langle c\rangle})$ [9], where d is the diameter of the filament and $\langle c\rangle = 0.05$–0.30). In this case, the axial friction force that acts on the filament surface is determined by the formula [8]

$$dP = A_\tau \Delta v_\tau \, ds, \tag{1.8}$$

where $A_\tau = \pi\, d\mu/[\langle r\rangle \ln(2\langle r\rangle/d)] = 2.1\pi\mu/\ln(0.952/\sqrt{\langle c\rangle}\,)$ and Δv_τ is the relative velocity of the longitudinal flow of the fluid about the filament.

Formulas (1.7) and (1.8) reflect exactly the linear dependence of the friction on the velocity for a laminar flow, though the numerical values of the coefficients A_τ and B_n can be refined.

The stationary plane velocity field of the fluid is characterized by the components $v_x(x, y)$ and $v_y(x, y)$. For an arbitrary point M at the filament, the velocity components have the form $\partial x/\partial t = \dot{x}$ and $\partial y/\partial t = \dot{y}$ (Fig.1b). The viscous-friction force is due to a certain lag of the filament behind the moving surrounding fluid. For example, in the direction of the x axis, the fluid velocity v_x exceeds the filament velocity $\dot{x}$ by the magnitude $v_x - \dot{x}$. Projecting the velocities onto the tangent and the normal to the filament axis, for the relative velocities we obtain the following expressions:

$$\Delta v_n = (v_x - \dot{x})\sin\varphi - (v_y - \dot{y})\cos\varphi, \quad \Delta v_\tau = (v_x - \dot{x})\cos\varphi + (v_y - \dot{y})\sin\varphi.$$

The derivatives with respect to t are dotted. As in (1.1), by replacing the trigonometric functions by the quantities x' and y', we have

$$\Delta v_n = (v_x - \dot{x})y' - (v_y - \dot{y})x', \quad \Delta v_\tau = (v_x - \dot{x})x' + (v_y - \dot{y})y'. \tag{1.9}$$

Considering jointly (1.3) and (1.5)–(1.9), we obtain a system that describes the nonstationary strain of a flexible filament:

$$\begin{gathered} N' + A_\tau[(v_x - \dot{x})x' + (v_y - \dot{y})y'] = 0, \\ B_n[(v_x - \dot{x})y' + (v_y - \dot{y})x'] - Ny''/x' = 0, \qquad (x')^2 + (y')^2 = 1. \end{gathered} \tag{1.10}$$

Equations (1.10) should be supplemented with the initial and boundary conditions:

$$\begin{aligned} t = 0: &\quad x = x_0(s), \quad y = y_0(s), \quad N = 0; \\ t > 0: &\quad s = \pm l, \quad N = 0, \quad y'' = 0. \end{aligned} \tag{1.11}$$

Where $2l$ is the filament length and $x_0(s)$, $y_0(s)$ are the parametric description of the initial shape of the filament. We reckon s from the middle of the filament: the positive and negative directions to the right and to the left, respectively (Fig. 1). According to (1.6) or the second equation in (1.10), the absence of the stretching force at the free ends of the filament is equivalent to the zero curvature of the filament's end $y'' = x'' = 0$ for $s = \pm l$. The initial tension is absent.

Now we analyze two rheological types of two-dimensional flow: pure shear and simple shear. In the case of pure shear, the velocity components $v_x(x, y)$ and $v_y(x, y)$ are determined by the relations [10]

$$v_x = g|\gamma|x, \qquad v_y = -g|\gamma|y. \tag{1.12}$$

Here $\gamma = \partial v_x/\partial x$ is the strain rate and $g = \operatorname{sign}\gamma$ is a parameter that describes the flow direction (the expansion along the x axis and the compression along the y axis occur for $g = -1$ and vise versa for $g = -1$).

In the case of simple shear flow (Couette flow), the stationary motion of a fluid between two unbounded planes which are parallel and equidistant from the x axis is considered; the upper plane moves with constant velocity in the positive direction of x, and the lower plane moves with the same velocity in the negative direction. A laminar regime of flow with a linear velocity profile is established in the gap. The x axis lies in the horizontal plane and remains fixed. In addition, one can consider that the x axis of the convective coordinate system is "frozen" into the middle layer of the fluid. For simple shear [10], the velocity components have the form

$$v_x = g|\gamma_-|y, \qquad v_y = 0. \tag{1.13}$$

Here $\gamma_- = \partial v_x/\partial y$ is the shear velocity and $g = \operatorname{sign}\gamma_-$ is a parameter that describes the direction of the flow: the upper plane moves from left to right and the lower plane moves from right to left for $g = 1$ and vise versa for $g = -1$.

2. Motion of a Filament under Pure-Shear Conditions. We introduce the following dimensionless variables and parameters:

$$\tau = t|\gamma|, \quad \{X, X_0, Y, Y_0, S\} = \{x, x_0, y, y_0, s\}/l, \quad N_+ = N/(A_\tau|\gamma|l^2), \quad E = A_\tau/B_n. \tag{2.1}$$

For the velocity components of the fluid (1.12), the problem (1.10), (1.11) has the following dimensionless form:

$$\begin{gathered} N_+' + (gX - \dot{X})X' - (gY + \dot{Y})Y' = 0, \\ (gX - \dot{X})Y'X' + (gY + \dot{Y})(X')^2 - EN_+Y'' = 0, \qquad (X')^2 + (Y')^2 = 1, \\ \tau = 0: \quad X = X_0(S), \quad Y = Y_0(S), \quad N_+ = 0, \\ \tau > 0: \quad S = 0: \; X = 0, \; Y = 0; \qquad S = 1: \; N_+ = 0, \; Y'' = 0. \end{gathered} \tag{2.2}$$

Hereinafter, the derivatives with respect to τ are dotted, and those with respect to S are primed. The boundary conditions in (2.2) are written in the centrosymmetric, initial configuration of the filament. The

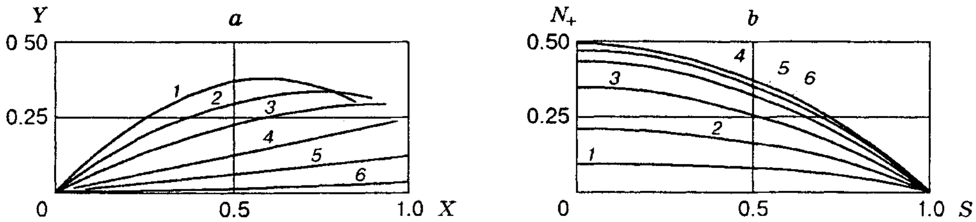

Fig. 2

middle of the filament is in the coordinate origin, and the relations $X(S) = -X(-S)$ and $Y(S) = -Y(-S)$ hold. Here, for the symmetric velocity fields of the fluid (1.12) and (1.13), the middles of the filament during its strain will always be at the origin of coordinates (the convective displacement of the filament is eliminated). In this case, it suffices to consider the evolution of the right half of the filament ($0 \leqslant S \leqslant 1$).

According to (2.1) and (2.2), the fluid viscosity determines tension but does not affect the evolution of the form. Other conditions being equal, the filament tension is proportional to the viscosity, the strain rate, and the length of the filament squared. The experimental data [1–3] support the amplification of the tearing strength of a fibrous filler as the viscosity of the medium, the strain rate [1], and the initial length of the filaments [2, 3] increase. The evolution of the form is influenced by the velocity field of the fluid, the relation between the friction forces E, and the initial configuration of the filament.

Problem (2.2) is analyzed numerically for the following mixing conditions of polycaproamide filaments with a rubber matrix: $d = 30$ μm, $2l = 10^{-2}$ m, $|\gamma| = 18$ sec^{-1}, $\mu = 10^5$ Pa $\cdot$ sec, $\rho = 1200$ kg/m^3, $\langle c\rangle = 0.05$, $\langle v\rangle \approx |\gamma|l/2$, Re $= 4.32 \cdot 10^{-8}$, and $E = 1.53$. Instead of the second equation in (2.2), we use the equation derived by discarding $\dot{X}$ from the first two equations:

$$EN_+Y'' + N_+'Y' - gY = \dot{Y}, \tag{2.3}$$

which contributed to the stability of the difference scheme. The solution was found on a difference rectangular grid where $\Delta S = 0.01$ and $\Delta\tau = 0.0025$. The tension N_+ was determined from the first equation in (2.2) by the left two-point sweeping (the "implicit right corner" scheme [11]), beginning with the point $S = 1$. From Eq. (2.3), the values of Y on the upper time layer were found by the left three-point sweeping (the Crank–Nicholson scheme). The coupling equation [the third equation in (2.2)] was used to determine X. The values of N_+, Y, and X on the upper time layer were refined by iterations.

Figure 2a and b shows the evolutions of the filament configuration and the tension distribution over the length for $\tau = 0$; 0.1; 0.2; 0.4; 0.8; and 1.6 (curves 1–6, respectively). As in Fig. 3, curve 1 in Fig. 2b was obtained for N_+ at the first step in time. The initial shape of the filament was described by the sinusoid where $Y_0 = a\sin(\omega X_0)$, $a = 0.4$, and $\omega = 2.7$. The flow direction is $g = 1$. As can be seen in Fig. 2, one can conventionally distinguish two periods in the evolution of a filament of arbitrary initial form. In the first period, the distribution of the stretching force along the filament length is inhomogeneous and even compression sites are possible. The curvature of the filament decreases to $Y'' = 0$. The evolution depends on the initial configuration. In the second period, the filament, remaining rectilinear, rotates about the point $X = Y = 0$ streamwise. The evolution does not depend on the initial configuration. The tension distribution is described by a parabola with a vertex at the point $S = 0$. As $\tau \to \infty$, the filament axis coincides with the streamline passing through the coordinate origin. For $g = 1$, the streamline coincides with the X axis (with the $g = -1$ axis for Y). After the second period terminates, the tension in the filament reaches the maximum $N_{\max} = N_{(S=0)} = 0.5A_\tau|\gamma|l^2$. Probably, precisely this moment is dangerous from the viewpoint of fracture of the filler with its poor tensile strength.

The filament evolution is reversible: if one changes the direction of the fluid flow (the sign of g is inverted), the filament will move to the initial position.

An analysis of problem (2.2) showed that the initially rectilinear filament preserves its rectilinear shape during evolution, and the tension distribution over the length is described by a parabolic dependence. Its evolution corresponds to the second period of evolution of a curvilinear filament. For a rectilinear filament, one can obtain an analytical solution of problem (2.2).

Let the angle of slope of a rectilinear filament to the X axis be described by the function $\varphi(\tau)$. The initial condition for the filament is

$$\varphi = \varphi_0, \qquad \tau = 0. \tag{2.4}$$

For the functions X and Y, we assume expressions that satisfy the coupling equation [the third equation in (2.2)]:

$$Y = S \sin\varphi, \qquad X = S \cos\varphi. \tag{2.5}$$

The tension distribution over the filament length is parabolic:

$$N_+ = \Psi(\tau)(1 - S^2). \tag{2.6}$$

Substituting (2.5) and (2.6) into the first two equations of (2.2), after simple transformations we obtain the equations for the functions $\Psi(\tau)$ and $\varphi(\tau)$:

$$\Psi = 0.5g \cos 2\varphi, \qquad \dot{\varphi} = -g \sin 2\varphi. \tag{2.7}$$

Integrating the second equation with allowance for the initial conditions (2.4), we find the angle of slope versus the time:

$$\varphi = \arctan[\tan\varphi_0 \exp(-2g\tau)]. \tag{2.8}$$

Thus, the evolution of a rectilinear filament is described by the equations

$$\begin{gathered} Y = S \sin\varphi, \qquad X = S \cos\varphi, \qquad N_+ = 0.5g(1 - S^2)\cos 2\varphi, \\ \varphi = \arctan[\tan\varphi_0 \exp(-2g\tau)]. \end{gathered} \tag{2.9}$$

According to (2.9), the filament tension is zero for $\varphi = \pi/4$: $N_+ = 0$. The stretching forces occur in the sector $|\varphi| < \pi/4$ for $g = 1$ or $\pi/2 > |\varphi| > \pi/4$ for $g = -1$. The stretching force reaches the maximum in the middle of the filament $S = 0$ for $\varphi = 0$ and $g = 1$ or for $|\varphi| = \pi/2$ and $g = -1$ and is $N_+ = 0.5$ (in dimensional form, $N_{\max} = N_{(S=0)} = 0.5A_\tau|\gamma|l^2$). Here the "effective longitudinal viscosity" of a system filled with filaments oriented in the expansion direction is maximal (exceeds the viscosity of the matrix). For the orientation of the filaments $|\varphi| = \pi/4$, their tension is zero, and the effective viscosity of the system is close to the viscosity of the matrix.

Solution (2.9) describes the evolution for any orientation of the filament even in the presence of compressing forces in it. However, a numerical analysis of problem (2.2) showed that in the case of compressing forces in the filament, the computational scheme loses its stability. Here, the middle of the filament ($|S| \leqslant 0.2$), where the compressing forces are the most intense, acquires a sawtooth-like shape with a period approximately equal to $2\Delta S$ immediately before the difference scheme loses its stability. The strong fracture of the glass filaments of the filler is probably caused by their bending upon longitudinal compression in the middle part of the filament (see [1, 3]).

To find the bounds of applicability of solution (2.9) and of stability of Eqs. (2.2), we have to analyze the stability of the solution against small perturbations. The stability problem is formulated as follows. Let the position of a rectilinear filament be characterized by the angle φ at a certain moment of time. We set the increments of the functions X, Y, and N_+. It is required to clarify whether a perturbation that is caused by these increments will increase unboundedly or damp.

We now study a "rapid" instability at the initial moment of filament evolution, assuming that the perturbations grow so rapidly that the undisturbed motion can be considered "frozen." The hydromechanical system considered has no "memory;" therefore, any intermediate position of the rectilinear filament, which is characterized by the angle φ, may be considered initial. Therefore, in analyzing the stability, the angle φ in the equations is treated as a parameter.

To solve (2.9), we introduce small perturbations in the filament form α and β and of tension Υ:

$$X = S\cos\varphi + \alpha(\tau, S), \quad Y = S\sin\varphi + \beta(\tau, S),$$
$$N_+ = 0.5g(1 - S^2)\cos 2\varphi + \Upsilon(\tau, S), \quad |\alpha, \beta, \Upsilon| \ll 1. \tag{2.10}$$

Substituting (2.10) into (2.2) and linearizing, we obtain the following equations for deviations:

$$\Upsilon' + (g\alpha - \dot\alpha)\cos\varphi + \alpha' gS\cos\varphi\cos 2\varphi - (g\beta + \dot\beta)\sin\varphi + \beta' gS\sin\varphi\cos 2\varphi = 0,$$
$$0.5Eg(1 - S^2)\beta''\cos 2\varphi + \Upsilon'\sin\varphi - \beta' gS\cos 2\varphi - g\beta - \dot\beta = 0, \quad \alpha' + \beta'\tan\varphi = 0. \tag{2.11}$$

Here relation (2.7) was used for $\dot\varphi$.

A stability analysis is reduced to a study of the eigenvalue problem for system (2.11) with the boundary conditions

$$S = 0: \quad \alpha = \beta = 0, \quad \Upsilon = \Upsilon_0, \quad \beta' = \beta_0'; \qquad S = 1: \quad \Upsilon = 0, \quad \beta'' = 0. \tag{2.12}$$

We present perturbations in the form $\{\alpha, \beta, \Upsilon\} = \{A, B, C\}\exp(\lambda\tau)$, where $A(S)$, $B(S)$, and $C(S)$ are eigenfunctions of the problem and λ is the eigenvalue. In correspondence with (2.11) and (2.12), for the eigenfunctions we obtain the problem

$$C' + A(g - \lambda)\cos\varphi + A' gS\cos\varphi\cos 2\varphi - B(g + \lambda)\sin\varphi + B' gS\sin\varphi\cos 2\varphi = 0,$$
$$0.5Eg(1 - S^2)B''\cos 2\varphi + C'\sin\varphi - B' gS\cos 2\varphi - (g + \lambda)B = 0, \quad A' + B'\tan\varphi = 0,$$
$$S = 0: \quad A = B = 0, \quad C = C_0, \quad B' = B_0'; \qquad S = 1: \quad C = 0, \quad B'' = 0.$$

After integration of the last equation with allowance for the boundary conditions, we obtain $A = -B\tan\varphi$. The functions A and B are linearly dependent, which allows us to exclude the function A and write the following equations for the functions B and C:

$$0.5Eg(1 - S^2)B''\cos 2\varphi - B' gS\cos 2\varphi - (g\cos 2\varphi + \lambda)B = 0, \quad C' = 2Bg\sin\varphi,$$
$$S = 0: \quad B = 0, \quad C = C_0, \quad B' = B_0'; \qquad S = 1: \quad C = 0, \quad B'' = 0. \tag{2.13}$$

The eigenfunctions are defined with accuracy up to an arbitrary factor; therefore, it is possible to set $B_0' = 1$ without loss of generality.

We search for the solution of problem (2.13) in the Gauss plane. To do this, we introduce the following notation: $\lambda = \lambda_r + i\lambda_i$, $B = B_r + iB_i$, and $C = C_r + iC_i$. For the functions and the eigenvalue, we have $B_r = B_i = S$, $C_r = C_i = g(S^2 - 1)\sin\varphi$, $\lambda_r = -2g\cos 2\varphi$, and $\lambda_i = 0$.

Since $\lambda_i = 0$, there are no oscillations in the system. The identity of the dependence of the eigenvalue λ_r and the filament tension N_+ in (2.9) on φ is noteworthy. Therefore, the stable-strain area corresponds to the stretching-force region ($N_+ > 0$). For $g = 1$ and $|\varphi| < \pi/4$, the flow is stable ($\lambda_r < 0$). The values of the parameters $\varphi = \pi/4$, $\lambda_r = 0$, and $N_+ = 0$ correspond to the neutral stability of the filament. For $g = 1$ and $\pi/2 > |\varphi| > \pi/4$, the perturbations increase unboundedly ($\lambda_r > 0$), and the filament motion is unstable. Indeed, the loss of stability of the difference scheme was observed in this region for Eq. (2.2). Thus, the solution for a rectilinear filament (2.9) holds in the area $|\varphi| < \pi/4$ for $g = 1$ and in the region $\pi/2 > |\varphi| > \pi/4$ for $g = -1$.

3. Simple Shear (Couette flow). Substituting the velocity components (1.13) into Eqs. (1.10) and (1.11) in dimensionless variables (2.1), we obtain the following problem:

$$N_+' + (gY - \dot X)X' - \dot Y Y' = 0,$$
$$(gY - \dot X)Y'X' + \dot Y(X')^2 - EN_+Y'' = 0, \qquad (X')^2 + (Y')^2 = 1,$$
$$\tau = 0: \qquad X = X_0(S), \quad Y = Y_0(S), \quad N_+ = 0, \tag{3.1}$$

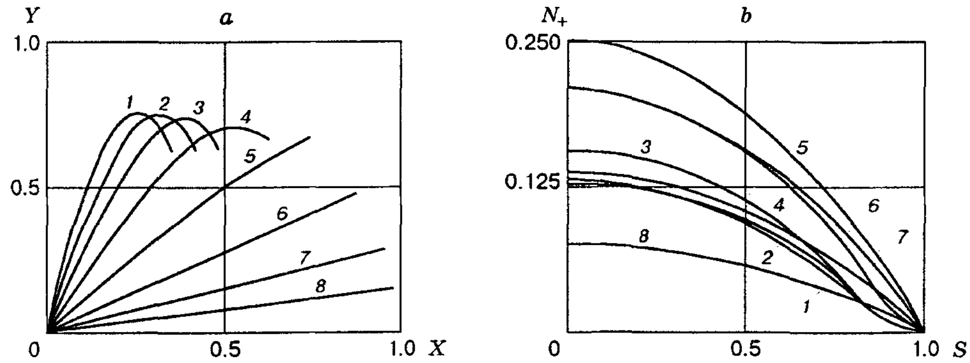

Fig. 3

$$\tau > 0: \quad S = 0: \quad X = 0, \quad Y = 0; \qquad S = 1: \quad N+ = 0, \quad Y'' = 0.$$

Here, the shear velocity $|\gamma_-|$ is used in the expressions for τ and N_+.

Figure 3a and b shows the evolution of the form and tension of a sinusoidal filament, respectively, under the simple-shear conditions for $\tau = 0$, 0.1, 0.2, 0.4, 0.8, 1.6, 3.2, and 6.4 (curves 1–8). The calculations were performed under the conditions indicated in Sec. 2 but for $a = 0.75$, $\omega = 6$, and $g = 1$. As in Fig. 2, one can see in Fig. 3, that the filament motion can be divided into two periods. The curvature of the filament decreases, evolving to a rectilinear shape in the first period and the rectilinear filament rotates in the flow direction in the second period.

The character of tension variation differs greatly from the case of pure shear. In the first period, the tension increases, and its distribution gradually takes a parabolic form. In the second period ($\tau > 0.8$), the tension, preserving the parabolic character of the distribution over the filament length, decreases to zero. (It is shown below that the maximum tension occurs for the angle of slope of the filament $\varphi = \pi/4$.) It follows from an analysis that the initially rectilinear filament conserves its shape during evolution.

For a rectilinear filament, an analytical solution of problem (3.1) is possible. The course of the solution is similar to that given in Sec. 2. We seek a solution in the form (2.5), (2.6). Substituting these expressions into (3.1), we obtain the equations for $\Psi(\tau)$ and $\varphi(\tau)$

$$\Psi = 0.25g \sin 2\varphi, \qquad \dot{\varphi} = -g \sin^2 \varphi.$$

With allowance for conditions (2.4), the solution of the second equation has the form

$$\tau g = \cot \varphi - \cot \varphi_0.$$

Thus, under the simple-shear conditions the evolution of a rectilinear filament is described by the equations

$$Y = S \sin \varphi, \qquad X = S \cos \varphi, \qquad N_+ = 0.25g(1 - S^2) \sin 2\varphi,$$
$$\varphi = \arctan [\tan \varphi_0 / (1 + g\tau \tan \varphi_0)]. \tag{3.2}$$

Here, the condition $N_+ > 0$ should be satisfied in the sector $-\pi/2 < \varphi < \pi/2$; for this, the equality sign $\varphi = g$, which follows from the singularity of the expression for $\varphi(\tau)$ in (3.2) (g and $\tan \varphi_0$ should have the same sign), should strictly hold. It follows from the expression for N_+ in (3.2) that the maximum tension in the filament occurs for $\varphi = \pi/4$ and is $N_+ = 0.25$ for $S = 0$ ($N_{\max} = 0.25 A_\tau |\gamma_-| l^2$), i.e., it is a factor of two smaller than that for pure shear. The significant distinction from pure shear consists of the fact that upon termination of the second period, when the filament axis coincides with the streamline ($g = 1$, $Y = 0$, $\tau = \infty$, and $\varphi = 0$), the tension is equal to zero ($N_+ = 0$). A comparison of the functions $\varphi(\tau)$ in (2.8) and (3.2) showed that the rotational velocity of the filament in the flow direction for simple shear is smaller than that for pure shear.

Therefore, the effect of the filament "orientation" during pure shear is expressed more distinctly. However, during pure shear (approximately $4\mu\gamma^2$) the specific energy expenditures necessary for viscous-fluid strain, which are characterized by a dissipation function, are greater than in the case of pure shear (of the order $\mu\gamma_-^2$).

For $\operatorname{sign}\varphi = -g$, compressing forces appear in the filament, and the difference scheme of problem (3.1) becomes unstable. The stability is lost even if the nonrectilinear segment of the filament is subjected to compressing load ($N_+ < 0$).

The stability of a rectilinear filament against small perturbations was investigated. By analogy with (2.10), we represent perturbations as follows:

$$X = S\cos\varphi + \alpha(\tau, S), \qquad Y = S\sin\varphi + \beta(\tau, S),$$

$$N_+ = 0.25g(1 - S^2)\sin 2\varphi + \Upsilon(\tau, S), \qquad |\alpha, \beta, \Upsilon| \ll 1.$$

The linearized perturbation equations have the form

$$\Upsilon' + (g\beta - \dot{\alpha})\cos\varphi - \dot{\beta}\sin\varphi = 0,$$

$$0.25Eg(1 - S^2)\beta''\sin 2\varphi + \Upsilon'\sin\varphi - 0.5\beta' gS\sin 2\varphi - \dot{\beta} = 0, \qquad \alpha' + \beta'\tan\varphi = 0,$$

$$\tau > 0: \quad S = 0: \;\; \alpha = \beta = 0, \;\; \Upsilon = \Upsilon_0, \;\; \beta' = \beta_0'; \qquad S = 1: \;\; \Upsilon = 0, \;\; \beta'' = 0.$$

We introduce the eigenfunctions of the problem $A(S)$, $B(S)$, and $C(S)$ and the eigenvalue λ: $\{\alpha, \beta, \Upsilon\} = \{A, B, C\}\exp(\lambda\tau)$. Taking into account the linear dependence of the functions A and B (see Sec. 2), for the functions B and C we have the equations

$$C' + gB\cos\varphi = 0, \quad 0.25Eg(1 - S^2)B''\sin 2\varphi - 0.5B'gS\sin 2\varphi - (0.5g\,\sin 2\varphi + \lambda)B = 0,$$

$$S = 0: \;\; B = 0, \;\; C = C_0, \;\; B' = 1; \qquad S = 1: \;\; C = 0, \;\; B'' = 0.$$

The solution of this problem in the complex plane has the form $B_r = B_i = S$, $C_r = C_i = -0.5g(1 - S^2)\cos\varphi$, $\lambda_r = -g\,\sin 2\varphi$, and $\lambda_i = 0$. Oscillations are absent ($\lambda_i = 0$). The dependence $\lambda_r(\varphi)$ is similar to the dependence $N_+(\varphi)$ in (3.2); therefore, if the filament undergoes the stretching forces $N_+ > 0$ and $\lambda_r < 0$, its motion is stable. For $|\varphi| < \pi/2$, the stability condition $\lambda_r < 0$ is satisfied if $-g\,\operatorname{sign}\,\varphi < 0$. The neutral stability occurs in the horizontal position of the filament ($\lambda_r = 0$, $\varphi = 0$, and $N_+ = 0$).

The static equilibrium of the filament coincides with the neutral stability. Therefore, in real conditions the filament orientation is unstable, because the residual flexural elasticity of the filament or the fluid-perturbation velocity can displace the filament to the region $\varphi < 0$ (for $g = 1$), which eventually leads to rotation of the filament about the center being in its middle. The angular rotational velocity is irregular: it reaches the maximum for $|\varphi| = \pi/2$ and the minimum in the neighborhood of $\varphi = 0$. Thus, the simple shear flow does not ensure the stable orientation of an isolated finite-length filament.

Under the simple-shear conditions, the greatest viscosity of the filled system corresponds to the filament orientation $|\varphi| = \pi/4$ when the tension in them is maximal.

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